Senior Project Mid-Term Report:

"Investigating a Correlation between Minimal Surfaces and Relativistic String Dynamics"

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Progress Update

The most essential and most illuminating work completed so far in this project has been the analysis of two example string solution worldsheets. First, that of a linearly-rotating straight open string, and second, that of an initially static circular closed string. These are established [1] to correspond to two well-known Euclidean minimal surfaces, the helicoid and catenoid, respectively. I derived the equations of motion for both of these string initial conditions directly from the Nambu-Goto bosonic string action [2] (image from [3]):

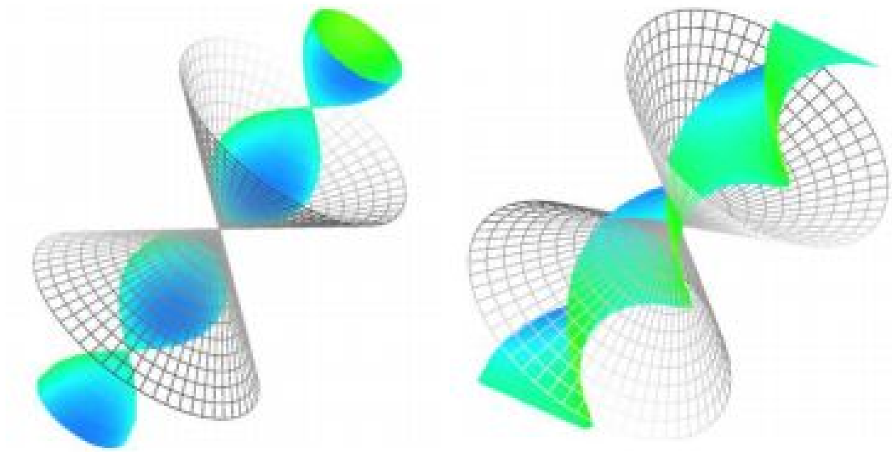
$$S = -T \int d^2\sigma \sqrt{-(\dot{X})^2 (X')^2 + (\dot{X} \cdot X')^2}\ .$$

where ' denotes a derivative with respect to σ and the superscript dot denotes a derivative with respect to τ. These derivations are dependent on a key feature of the Nambu-Goto action, its reparameterization invariance, which as explained in [3], is a gauge symmetry, and is essential to ensure that space and time are treated on equal footing as required by relativity. It is thus permissible to reformulate the above action in terms of chosen parameters σ and τ, which are selected in such a way that the string equations of motion are drastically simplified.

The helicoid worldsheet is expressed as: $X(t,\sigma) = (\frac{\sigma_1}{\pi} cos(\frac{\pi\sigma}{\sigma_1}) cos(\frac{\pi ct}{\sigma_1}), \frac{\sigma_1}{\pi} cos(\frac{\pi\sigma}{\sigma_1}) sin(\frac{\pi ct}{\sigma_1}), ct)$

while the catenoid is expressed as: $X(t,\sigma) = (R cos(\frac{2\pi\sigma}{\sigma_1}) cos(\frac{2\pi ct}{\sigma_1}), R sin(\frac{2\pi\sigma}{\sigma_1}) cos(\frac{2\pi ct}{\sigma_1}), ct)$

where σ_1 always refers to the parameter "length" of an open string, or identification length for a closed string, and the ct vector component defines one dimension of the surface as the direction of Minkowski time. Using the differential geometric techniques derived in [4] and [5], the mean curvature for the helicoid worldsheet surface was calculated to be 0, demonstrating that it is a minimal surface. However, when the same techniques were applied to the catenoid worldsheet, a positive definite equation for mean curvature appeared, oscillating as a function of time, which would seem to violate the minimal surface condition. Looking at plots of the catenoid and helicoid worldsheets in Minkowski space as provided by [6]:



The left figure, the catenoid, indeed describes the motion of a closed circular string, as is described in [1]. However, unlike the helicoid figure, the catenoid surface does not visually appear to be minimal; in fact, every point appears to have a positive curvature, just as the derived mean curvature equation would suggest. How, then, could such a string solution emerge from the minimization of an action principle which is by definition [1] proportional to area on the worldsheet?

The resolution to this conflict was found in the distinction between Euclidean and Lorentzian geometries. We humans are unavoidably trained in the mechanics of Euclidean space, where distance between two points increases as their positions in any single coordination direction become more different. However, 3-Minkowski space is described by a metric signature of <-,+,+> rather than Euclidean <+,+,+>. It is the resultant “invariant interval” $ds^2 = -c^2dt^2 + dx^2 + dy^2$, from which all relativistic properties of the strings under study are derived, that also leads to confusion with the appearance of minimal surfaces. I had attempted to study my worldsheets using Euclidean differential geometry, when only the geometry of the Lorentz space would enable the correct identification of minimal surface properties. I further realized that a transformation t→ i*τ would convert the periodic catenoid worldsheet to the more familiar expression for the catenoid surface in the form of hyperbolic cosine, by virtue of the identity cos(i*t) = cosh(t). Furthermore, and more generally, this transformation causes the interval expression $ds^2 = -c^2dt^2 + dx^2 + dy^2$ to become $ds^2 = c^2d\tau^2 + dx^2 + dy^2$, describing an effective Euclidean geometry. This transformation is well-known as the Wick transformation, and as mentioned in [6], enables Lorentzian problems to be expressed in a Euclidean geometry, and vice-versa. However, it is noteworthy that this simple trick is limited to use in flat geometries; more subtle techniques would have to be used for the study of curved spaces such as AdS space.

Technical details aside, the essential result of these studies was that <u>all</u> string worldsheets, which are by definition minimal surfaces in Minkowski space, can be transformed into a minimal surface in Euclidean space by the Wick transformation. This is confirmed in [6]. Two of my original goals for this project, which involved classifying string solutions by whether or not they corresponded to a Euclidean minimal surface, are therefore no longer in need of completion. However, the universality of this correspondence allows me to now pursue the discovery of novel string solutions and minimal surfaces, by applying techniques unique to both fields and transforming between Euclidean surfaces and their Lorentzian equivalents. My goals below have been updated to reflect this change.

Updated Goals

-- I will examine links between physical quantities such as momentum flow along the worldsheet and geometry quantities such as curvature on the corresponding Euclidean minimal surface. Intuition suggests that significant correlations will exist between these quantities, and this information may then be applied to future surfaces under study. (1-2 weeks)

-- The “light-cone gauge” is a coordinate transformation which allows the string equations of motion to be more easily solved, in a general form of mode expansion [1]. I will try to use these more general string solutions to generate Euclidean minimal surfaces. (2-3 weeks)

-- The Weierstrass-Enneper parameterization [7] is a solution technique for the equations that define minimal surfaces, wherein any minimal surface is shown to be described uniquely by two ‘parameter’ functions. Simple examples of these functions have been used to generate many interesting minimal surfaces; I will attempt to apply this technique to the generation of new relativistic string solutions. (2-3 weeks)

By pursuing these goals for the remainder of the semester, I intend to maximize what is achievable in the remaining weeks of the semester.

References

Text:

- [2] Zwiebach, Barton. A First Course in String Theory. New York: Cambridge UP, 2004. Print. Chapters 1-10
- [4] Kühnel, Wolfgang. Differential Geometry: Curves - Surfaces - Manifolds. Providence, RI: American Mathematical Society, 2006. Print.
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- [1] Correspondence. Prof. Matthew Kleban. NYU. 2016.
- [3] Tong, David. "The Relativistic String." String Theory Lecture Notes. Cambridge University, n.d. Web. 21 Sept. 2016.
- [6] Lee, Sungwook. "Weierstrass Representation for Minimal Surfaces in Minkowski 3-space." Arxiv.org. N.p., 29 Aug. 2006. Web. 13 Oct. 2016.
- [7] Kilchrist, Myla, and Dave Packard. "The Weierstrass-Enneper Representations." Dynamics at the Horsetooth. Colorado State University, Spring 2012. Web. 13 Oct. 2016.